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On Vanishing Fractions. By Jared Mansfield, Professor of the Military Academy at Washington.—Read, May 17th, 1816.

THE numerical value of an algebraic expression, is not to be estimated, in all cases, from the value of the terms which compose it, independently or separately considered; or from the coalesced value of any part of them, but from the joint effect of the whole. For sometimes one or more of the terms vanish, or become infinite; and, therefore, as these are non-entities, or inconceivable by our understanding, we may hastily conclude the whole to be impossible, or not susceptible of management: whereas, by some contrary operation implied in the general expression, the evanescence, or infinity, may be destroyed. This is more particularly the case with those functions of a variable quantity, denominated *Vanishing Fractions.* The principles necessary for a developement and right understanding of this subject, are analogous to those employed in explaining the first elements of algebra, in respect to the use of the negative sign. As the abstract consideration of negative quantities lead to absurdity and erroneous conclusion, so does that of quantities which are supposed to be nothing, infinite, or more than infinite.

In order to clear this subject of its difficulties, it may be well to observe, generally, that algebraic expressions, or func-

tions of any quantity are a combination of ratios, either arithmetical, or geometrical. The effect of these on the function, does not depend on the absolute magnitude of the terms, but on their relative, or comparative value; for no variation of ratio arises from the variation of the magnitudes of quantities compared, under the same circumstances and considerations: thus the ratio of the diagonal of a square to its side, or of the diameter of a circle to the circumference, is the same, whatever the magnitude of those figures may be. Indeed the reasonings of mathematics consist altogether in the investigations of *relative magnitude*, or of relations generally, in which the consideration of absolute magnitude is excluded, and is only resorted to as a unit, or standard of numerical computation.

When a ratio is formed by a comparison, either arithmetical or geometrical, of two equal quantities, the function is not affected by it, whether they be supposed finite, infinite, or nothing. This inference is drawn, not from any consideration of quantities infinitely great or infinitely small, but rather from the reverse, viz. that in all possible states of their existence, there cannot, by hypothesis, be any ratio of *inequality;* there cannot, therefore, be any effect produced by such a ratio, in any state of the existence of the terms, and it is the same when they become non-entities. This argument might be stated logically, according to the *reductio ad absurdum.* Again, whatever the magnitude of the terms compared may be, if their ratio, or the relative value of one to the other, be great, or little, the function will be proportionally increased or diminished; but this, as has been already observed, must be estimated from the aggregate effect of all the ratios included in the function.

If there be some ratios of majority, and some of minority, these, like positive and negative quantities in algebra, have contrary effects, and where there is such a combination, the negative, or even impossible quantities by themselves, are not therefore to be considered as impossible in their effects on one another. A negative quantity, or an impossible expression in algebra by itself, is unmanageable, and not susceptible of

ratiocination; but in composition with others of the same kind, it may be rendered possible, or made to vanish. Thus the value of $x+\sqrt{-a}$, or of $\sqrt{-a}$, in itself is imposssible, and inconceivable by our understandings, but that of $x+\overline{\sqrt{-a}\times\sqrt{-a}}$ $=x-a$, is a pure algebraic quantity, or function of x, whose effect is simple and obvious. So likewise $x\sqrt{a-b}$, when b is greater than a, is an impossible function of x, but that of $x\frac{\sqrt{a-b}}{\sqrt{a-b}}$, in the same circumstance of b, is assignable, being equal to x; or in the ratio of 1 to $\sqrt{a-b}$, the numerator, whatever its value may be, is destroyed, by an equal and contrary ratio in the denominator, or of $\sqrt{a-b}$ to 1, or $\frac{\sqrt{a-b}}{1}\times\frac{1}{\sqrt{a-b}}=1$. In order, therefore, to estimate the true value of any function, we must resolve it into all the ratios of which it is composed, and if any of them be impossible by themselves, before we conclude, that the whole is such, we must ascertain, whether, as in common algebra, the impossibles may not destroy one another, so as to produce altogether no effect in the function. This will be best illustrated by examples, and from them may be derived all those rules, which have been considered not merely as mysterious, but absurd.

It is obvious, that an arithmetical ratio of equality as in the simple function of x, $x\pm\overline{a-b}$, when $a=b$, produces no effect, or that its value is $x\pm o=x$: Let $x\frac{a}{b}$, be another function of x, involving the geometrical ratio $\frac{a}{b}$, this also, when it is a ratio of equality, or $a=b$, will produce no effect, although it be expounded by 1, unity in a geometrical, being equivalent to 0 in an arithmetical ratio; for $\frac{a}{b}$ involves two other ratios, viz. that of $1:a$ and $b:1$, which, when $a=b$, are reciprocals; or as much as unity is increased by one, it is diminished by the other, and therefore their compounded value is unity. If the ratio of 1 to a, be infinitely great in the antecedent, or a

be nothing, this is an infinite ratio of minority in the consequent, which by itself, would cause $a \times x$ to become nothing; also the other ratio, the reciprocal of the former, ($\frac{1}{b}$ or $\frac{1}{a}$) by itself, would cause x to be infinitely great, and by estimating the ratios thus separately, we find them vanishing or infinite, and of course out of the limits of our faculties. It is from such a process, that the unskilful have found difficulties, which they charge to the mysterious nature of the subject, and the unintelligible doctrine of mathematicians concerning infinities. Let now this quantity x, which has been put out of existence by bad management, be restored to its function, and remain unmolested, until the balance of powers, which is to establish its weight and consequence, be ascertained. The first ratio, viz. that of 1 to 0 is $\frac{1}{0}$, the second is its reciprocal or $\frac{0}{1}$, and these compounded make $\frac{1}{1}=1$, and, therefore, the true value of $x\frac{a}{b}$, in the circumstance of $\frac{a}{0}$ being $=\frac{o}{0}$, is $x\frac{0}{0}=x\times 1=x$.

The argument in words is this; the first ratio is an infinite ratio of minority in the consequent, the other is an infinite ratio of majority in the same. These two compounded, constitute a ratio of equality, which is numerically expressed by a unit.

From the preceding observations, one of the results of mathematics on this subject may be derived, viz. *that nothing divided by nothing is equal to unity;* or, *that unity is a mean proportional between nothing and infinity; also*, *that* $\frac{x \text{ infinite}}{x \text{ infinite}} = 1.$

When compounded expressions are found in the numerator and denominator of the fraction, there are oftentimes ratios

included in it, which do not so readily appear, and its true value, in consequence, is liable to be mistaken in the circumstance of one or more of the ratios vanishing. Functions of this kind are those, which have received the appropriate denomination of *Vanishing Fractions.*

Let $\frac{a-x}{b-y}$ be such a fraction, it is evident from what has been before observed, that if the numerator and denominator were equal, then, one being a direct and the other a reciprocal, and equal ratio, these two would destroy one another's effect, and the result would be equal to unity; but in the circumstance, when $x=a$, and $y=b$, it will be $a:b::x:y$, and by alternation and division, $a-x:b-y::x:y::a:b$, whence there are found three ratios, besides the arithmeticals $a-x$ and $b-y$, in the expression $\frac{a-x}{b-y}$, viz. $1:\overline{a-x}$, $\overline{b-y}:1$, and and this last $b-y$ compared with the former, $\overline{a-x}$, or that of $b:a$. The two former combined are equal to 1, and all conjoined equal $1\times\frac{a}{b}$. Hence it appears, that the ratio of $a-x$ to $b-y$ does not vanish, because the coalesced values of those quantities vanish; for that ratio is, in that case, the ratio of the terms $a:b$, or $x:y$, and it is only one of the ratios, viz. the arithmetical $a-x$, or $b-y$, which really vanishes. Again, let $\frac{a-x}{b-y}=\frac{a}{b}$ be multiplied by $a+x$, or increased by another ratio, as $1:\overline{a+x}$, and we shall have $\frac{a^2-x^2}{b-y}=\frac{a}{b}\times a+x=\frac{a}{b}\times 2a$, the true value of the fraction when $x=a$, and $b=y$. If $a=b$, and $x=y$, the expression becomes $\frac{a^2-x^2}{a-x}=\frac{a-x}{a-x}\times\overline{a+x}=2a$, when vanishing. Also, $\frac{a^2x-x^3}{a-x}$, or $\frac{a^3-x^3}{a-x}$, in the same circumstance $=3a$, and if $a=1$, then $\frac{1-x^2}{1-x}=2$, $\frac{1-x^3}{1-x}=3$, &c.

From which, it is manifest, that the geometrical ratio of $\overline{1-x}$ to $\overline{1-x^2}$ is real and determinate, though the arithmetical ratios, or the coalesced values of the terms of those quantities vanish, or $= o$. Moreover, those quantities have a determinate ratio, when they are negative in their combined values, or in effect less than nothing: for when in the fraction $\frac{1-x^2}{1-x}=\frac{1-x}{1-x}=1\times\overline{1+x}$, the value of x is greater than 1, $\overline{1-x}$ is negative, and the value of the fraction is $1\times\overline{1+x}$, or greater than before when vanishing. Suppose $x=2$, then $1\times\overline{1+x}=3$, &c. The value, therefore, of the fraction $\frac{1-x^2}{1-x}$, increases as x increases; when $x=0$, it is equal to $\frac{1}{1}$ or 1, when $x=1$, it is equal to 2, when it exceeds 1, or the coalesced terms of numerator and denominator are negative, it exceeds 2, and increases continually as x increases. This being the case, viz. the law being established, that the value of the fraction $\frac{1-x^2}{1-x}$, increases from unity as x increases from nothing, and so continues to increase; it would be a contradiction to this law, that when $x=1$, or had a real and determinate value, the fraction should be equal to nothing, or have no assignable value.

The illusion; which has prevailed on this subject, arises from the idea of the impossibility of any geometrical ratio existing between two or more terms of an arithmetical series, which taken together are equal to nothing, and other similar terms of such a series; whereas, it can be shown that the former ratio does not depend on the aggregate of the arithmetical ratios; for the terms themselves do not vanish with their differences, and are therefore susceptible of a comparison with the other terms, and this constitutes the geometrical ratio. This will be evident from the following example. Let $\frac{2rx-x^2}{2r-x}$, be another fraction, of which the numerator is the equation of the circle, where r is radius, x the abscissa, and

$\sqrt{\overline{2rx-x^2}}$ an ordinate; when $x=2r$, or the abscissa = the diameter, the aggregate or coalesced value of the numerator $\overline{2rx-x^2}=4r^2-4r^2=0$, or the ordinate vanishes, when the abscissa is equal to the diameter, which is also evident from the construction of the circle. Also the denominator $2r-x$, or the difference between the diameter and abscissa = 0. Now, though the arithmeticals $2rx-x^2$, $2r-x$, vanish, their geometrical ratio is not affected by the evanescence; for the ratio of the first terms, viz. $2r:2rx$, is that of 1 to x, and that of the second, or $-x:-x$ is the same, whence by composition $2r-x:2rx-x^2::1:x$, or $\dfrac{2rx-x^2}{2r-x}=\dfrac{2r-x}{2r-x}\times x=1\times x=2r$. If while the radius of the circle remains the same, we suppose $x=y$, then $\dfrac{2rx-x^2}{2r-x}=\dfrac{2ry-y^2}{2y-y}=2r$, and $\overline{2r-x}$ (the evanescent versed sine) : $\overline{2rx-x^2}$ the square of half the chord) : : $\overline{2y-y}$ (another evanescent versed sine) : $\overline{2ry-y^2}$ (the square of one half its corresponding chord); whence the *versed sines when vanishing are as the squares of their chords.*

It is from such considerations of the different ratios, which obtain among functions when vanishing, or when their aggregate value is nothing, that the different degrees of curvature of any curve, or the comparison of curvatures of different curves is susceptible of determination; and as on this depend the higher geometry, and the laws of centripetal forces, it may be proper to illustrate this doctrine by other examples.

Let a be to b, in any ratio of minority, or majority, then $\dfrac{b}{a}\times\overline{ax-x^2}=y^2$, is an equation of the ellipsis, in which the squares of the ordinates have the same value as in the circle, but, increased or diminished in the ratio of a to b, and while this is finite, the whole becomes equal to nothing in the same circumstance as before, when $\overline{ax-x^2}=y^2$ denoted a circle. But if a, which represents the diameter, becomes infinitely great $\dfrac{b}{a}\times\overline{ax-x^2}=\dfrac{bax}{a}-\dfrac{b}{a}x^2=bx=y^2$; because $\dfrac{a}{a}=1$, and $\dfrac{b}{a}=0$. If $a=0$,

then $\frac{a}{a}=1$, and $\frac{bax}{a}=bx$; $\frac{b}{a}x^2=x^2$ infinitely great $=y^2$; or since adding or subtracting a finite quantity to or from an infinite quantity, has no effect, $x^2=y^2$, whence x and y are equal straight lines infinitely extended. If a be finite, and x indefinitely small in comparison of a, in that circumstance, we shall have $bx=y^2$, which is the same equation as before, viz. that of a parabola, whose parameter is b: from which it appears, that an ellipsis and parabola having equal parameters, have their nascent ordinates, or nascent arcs equal, or their curvature is equal, and that this is proportional to their parameters. If $b=a$, or the parameter be made equal to the diameter, $\frac{b}{a}\times\overline{ax-x^2}=y^2$ becomes the equation of the circle, and x being indefinitely small, it will be $ax=y^2$, the same as before in the ellipsis and parabola; which shows that a circle whose diameter is equal to the parameter of either of those curves, has the same curvature, or becomes the osculatory circle. In the general expression for the ellipsis $\frac{b}{a}\overline{ax-x^2}=y^2$, when $a=$ the transverse axis, is infinite or nothing, the ordinate is finite or nothing, but when $a=b$, or the expression becomes $\overline{ax-x^2}=y^2$, which is the equation of the circle; then, when the diameter a, is infinite or nothing, the ordinate is infinite or nothing, whatever x, or the abscissa may be. Whence the ordinates corresponding with any finite abscissa in an infinite ellipsis or parabola are finite, but in an infinite circle they are infinite, or the curve becomes a straight line or nothing.

Let x, or the abscissa of the curve, from first being $=0$, become negative, or pass from affirmation to negation, which in geomety implies, that it passes to the other side, from whence it commenced its existence; the equation of the curve will then become $\frac{b}{a}ax+x^2=y^2$. This exceeds the equation of the parabola $ax=y^2$ by x^2, and that of the ellipsis falls short as much; from which it is obvious, that the curve is an hy-

perbola, whose ordinates fall on the other side of the diameter of the circle, with which it is compared. As the expression $ax+x^2$, is less than the square of the binomial $\frac{a}{2}+x$, by $\frac{a^2}{4}$ the locus of the latter, which is a straight line, will exceed that of the curve, always by the same quantity $\frac{a^2}{4}$; whence, if $\overline{\tfrac{1}{2}a+x}^2 = z^2$, then $z^2-y^2=\frac{a^2}{4}$, or $\overline{z+y}\times\overline{z-y}=\frac{a^2}{4}$, and in the general expression, this difference becomes $z^2-y^2=\frac{b}{a}\times\frac{a^2}{4}=\frac{ba}{4}$, which, putting $c=\frac{1}{2}$ the conjugate diameter, becomes $z^2-y^2=c^2$. For $\sqrt{ba}=$ conjugate diameter, and $\frac{ba}{4}=\frac{\sqrt{ba}}{2}=c$ squared; or *square of half the conjugate equals the difference of the squares of the ordinates z and y.* As this difference is always equal to a finite quantity, the curve will always approach, but can never coincide with that right line. This, therefore, is an asymptote to the curve.

The equation of the curve being transformed from the axis to the asymptote, if instead of the difference of two variable quantities, z and y, we substitute that of one of them, and a given quantity, we shall arrive at the general equation of the hyperbola, by lines parallel to both the asymptotes; therefore, putting the ratio of $r-u:r$ equal to that of $z-y:c$, and that of $r:r+x$ equal to that of $c:z+y$, it will be $r-u:r::r:r+a$. This expression is that of a secant of an arc, of which r is the radius, and u the abscissa. If to the latter the secant of a quadrant be applied ordinately, the curve becomes *a figure of secants.*

Moreover, $\overline{r-u}$, being reciprocally as $r+x$, their rectangle will always be a given quantity $=r^2$, whereof the two sides are lines drawn parallel to the asymptotes from any points of the curve. As this distance never becomes equal to nothing, or the ratio of $\frac{r}{r-u}$ and $\frac{r+x}{r}$, never becomes infinite, the hy-

perbolic space, comprehended between the curve asymptote, will continue to increase *ad infinitum*, but the increase of the area will be uniform, being that of an arithmetical progression, or the reciprocals of two equal ratios; for $-u : r :: r : x$, and $ru+xu=rx-ux=1$. But while these spaces are in arithmetical, $\frac{x}{r-x}$, or $\frac{r+x}{r}$, are in geometrical progression, consequently the former are the logarithms of the latter, or of the natural number $r+x$, and they are of the Napierian kind, when the comparison of x and u is made the same with the quantity r.

But the most important application of these principles, is to the metaphysics, or fundamental principles of fluxions, or the science of differentials. This, however, cannot be comprised in the compass of the present paper, but is intended for the subject of another.